

A GRID FUNCTION FORMULATION OF A CLASS OF ILL-POSED PARABOLIC EQUATIONS

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In this paper, we will discuss the grid function formulation of a class of parabolic equations that depend upon a parameter u^+ . This class of parabolic equations is ill-posed forward in time, in the sense that they do not have solutions in the sense of distributions. Under the hypothesis that $0 < u^+ < +\infty$, the ill-posed problem has been studied by Plotnikov in [29], where he introduced a notion of solution in the sense of Young measures. If $u^+ = +\infty$, then the problem has no solution also in the class of Young measures; however, a notion of measure-valued solution has been given by Smarrazzo [32] and can be characterized as the sum of a Young measure and a Radon measure. In Section 1 we will introduce the class of ill-posed parabolic

equations 1.1 and recall the definition of the measure-valued solutions for these problems.

Despite the different notions of measure-valued solutions, that depend upon the value of u^+ , we will be able to give a unique grid function formulation for the class of ill-posed PDEs. In section 3 we will derive the grid function formulation of problem 1.1 from a discrete model of diffusion, then we will discuss its solutions. In section 4, we will show the relations between the solutions to the grid function formulation and the solutions to the original problem 1.1. In particular, if the solutions to the nonstandard problem are regular enough, they induce solutions to problem 1.1 that are coherent with the approaches of Plotnikov and Smarrazzo. In section 5 we will discuss the asymptotic behaviour of the grid solutions to problem 1.1 by studying the asymptotic behaviour of the solution to the grid function formulation. We will also give a positive answer to a conjecture by Smarrazzo on the coarsening of the solutions to problem 1.1 when $u^+ = +\infty$. The paper concludes with a brief discussion of some properties of the grid solution to problem 1.1 with Riemann initial data. In particular, by studying this initial value problem, we will show that the grid solution to problem 1.1 features a hysteresis loop, in agreement with the behaviour of the Young measure solution.

1. THE ILL-POSED PDE

Consider the Neumann initial value problem

$$(1.1) \quad \begin{cases} \partial_t u = \Delta \phi(u) & \text{in } \Omega \\ \frac{\partial \phi(u)}{\partial \nu} = 0 & \text{in } [0, T] \times \partial \Omega \\ u(0, x) = u_0(x) \end{cases}$$

where $\Omega \subseteq \mathbb{R}^k$ is an open, bounded, and connected set, $u : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ and where the following hypotheses over ϕ are assumed:

Hypotheses 1.1. *ϕ satisfies:*

- $\phi(x) \geq 0$ for all $x \geq 0$ and $\phi(0) = 0$;
- $\phi \in C^1(\mathbb{R})$;
- ϕ is non-monotone, i.e. there are u^-, u^+ with $0 < u^- < u^+ \leq +\infty$ such that $\phi'(u) > 0$ if $u \in (0, u^-) \cup (u^+, +\infty)$ and $\phi'(u) < 0$ for $u \in (u^-, u^+)$;
- if $u^+ = +\infty$, then $\lim_{x \rightarrow +\infty} \phi(x) = 0$.

It is well-known that, if $u_0 \in L^\infty(\Omega)$ and $\text{ess sup } v \leq u^-$ or $\text{ess inf } v \geq u^+$, then the dynamics described by equation 1.1 amount to a parabolic smoothing. The main feature of problem 1.1 is that it is ill-posed forward in time for u in the interval (u^-, u^+) : there are no weak solutions to problem 1.1 whenever $\|u_0\|_\infty > \phi(u^-)$ and, if we allow for measure-valued solutions, then solutions to problem 1.1 exist for any initial data but are not unique.

Forward-backward parabolic equations like 1.1 or the closely related

$$(1.2) \quad u_t = \operatorname{div} \phi(\nabla u)$$

with non-monotone ϕ have been used to describe various physical phenomena. Cubic-like functions with $u^+ < +\infty$ arise for instance in models of phase transitions: in this context, the function u represents the enthalpy and $\phi(u)$ the temperature distribution. Equation 1.1 can be deduced as a consequence of the Fourier law.

If $u^+ = +\infty$, then equation 1.1 has been used in models of the dynamics of aggregating populations both in a discrete approximation (see for instance Horstmann, Othmer and Painter [15] and Lizana and Padron [19]) and as a continuous diffusion approximation [27]. Equation 1.2 has been used to describe also shearing of granular media (see Witelski, Shaeffer and Shearer [34]). It is also noteworthy to mention that the Perona-Malik edge-enhancement algorithm via backward diffusion [28] is based on equation 1.2.

The hypothesis that ϕ is non-monotone is crucial both for the applications and for the description of the physical phenomena. For this reason, suitable approximations of initial value problems for equations 1.1 and 1.2 has been studied in a variety of ways. For a discussion of these approaches, we refer to [24] and to [32].

1.1. The Young measure solution in the case $u^+ < +\infty$. The most common approach to problem 1.1 and to problem 1.2 is to treat them as the limit of some sequence of approximating problems: the notions of solutions will accordingly depend on the chosen regularization.

In [29], Plotnikov studied problem 1.1 by means of the following Sobolev regularization:

$$(1.3) \quad \begin{cases} u_t = \Delta v & \text{in } \Omega \\ v = \phi(u) + \eta u_t \\ \frac{\partial v}{\partial \nu} = 0 & \text{in } [0, T] \times \partial\Omega \\ u(x) = u_0 & \text{in } \Omega. \end{cases}$$

with $\eta > 0$. The Neumann initial-boundary value problem for this regularized problem under the hypothesis that $u^+ < +\infty$ has been studied by Novick-Cohen and Pego [26]. In particular, Novick-Cohen and Pego proved that if ϕ is locally Lipschitz continuous and the initial data is $L^\infty(\Omega)$, then there exists a unique classical solution (u_η, v_η) to problem 1.3, with $u \in C^1([0, T], L^\infty(\Omega))$ and $v_\eta = \phi(u) + \eta(u_\eta)_t$, and the functions (u_η, v_η) satisfy the inequality

$$(1.4) \quad \int_0^T \int_\Omega G(u_\eta) \varphi_t - g(v_\eta) \nabla \varphi - g'(v_\eta) |\nabla v_\eta|^2 \varphi dx dt \geq 0$$

for all non-decreasing $g \in C^1(\mathbb{R})$ and with $G' = g$, and for all $\psi \in \mathcal{D}([0, T] \times \Omega)$ with $\psi(t, x) \geq 0$ for all $(t, x) \in [0, T] \times \Omega$. This inequality has the role of an entropy condition for the weak solutions of problem 1.1, in a sense made precise by Evans [7] and by Mascia, Terracina and Tesei [24].

Since the sequences $\{u_\eta\}_{\eta>0}$ and $\{v_\eta\}_{\eta>0}$ are uniformly bounded in $L^\infty([0, T] \times \Omega)$, they have a weak- \star limit $(u, v) \in L^\infty([0, T] \times \Omega)$ that satisfies equation

$$(1.5) \quad \begin{cases} u_t = \Delta v & \text{in } \Omega \\ \frac{\partial v}{\partial \nu} = 0 & \text{in } [0, T] \times \partial\Omega \\ u(x) = u_0 & \text{in } \Omega \end{cases}$$

in the weak sense, i.e. $u \in L^\infty([0, T] \times \Omega)$, $v \in L^\infty([0, T] \times \Omega) \cap L^2([0, T], H^1(\Omega))$ such that

$$(1.6) \quad \int_0^T \int_\Omega u \psi_t - \nabla v \cdot \nabla \psi dx dt + \int_\Omega u_0(x) \psi(0, x) dx = 0$$

for all $\psi \in C^1([0, T] \times \overline{\Omega})$ with $\psi(T, x) = 0$ for all $x \in \Omega$. However, since in general weak- \star convergence is not preserved by composition with a nonlinear function, we have no reason to expect that $v = \phi(u)$, so that the weak solution of 1.5 is not a weak solution of 1.1.

Thanks to the uniform bound on the L^∞ norm of $\{u_\eta\}_{\eta>0}$, Plotnikov in [29] showed that the sequence $\{u_\eta\}_{\eta>0}$ has a limit point ν in the space of Young measures, and ν can be interpreted as a weak solution to equation 1.1. Moreover, ν can be characterized as a superposition of at most three Dirac measures concentrated at the three branches of ϕ , and the Young measure ν and the function v defined by $v = \int_{\mathbb{R}} \phi(\tau) d\nu$ still satisfy the entropy inequality 1.4. This analysis suggests the following definition of weak solution to problem 1.1 in the sense of Young measures.

Definition 1.2. *An entropy Young measure solution of problem 1.1 consists of functions $u, v, \lambda_i \in L^\infty([0, T] \times \Omega)$, $1 \leq i \leq 3$, satisfying the conditions:*

- (1) $\lambda_i \geq 0$, $\sum_{i=1}^3 \lambda_i = 1$, and $\lambda_1(x) = 1$ if $v(x) < \phi(u^+)$, $\lambda_3(x) = 1$ if $v(x) > \phi(u^-)$;
- (2) $v \in L^2([0, T], H^1(\Omega))$ and $u = \sum_{i=1}^3 \lambda_i S_i(v)$, where $S_i(v)$ are defined as follows:

$$\begin{aligned} S_1 &: (-\infty, \phi(u^-)] \rightarrow (-\infty, u^-], \\ S_2 &: (\phi(u^+), \phi(u^-)) \rightarrow (u^-, u^+), \\ S_3 &: [\phi(u^+), +\infty) \rightarrow [u^+, +\infty), \end{aligned}$$

and, for all i , $u = S_i(v)$ iff $v = \phi(u)$;

- (3) $u_t = \Delta v$ in the weak sense, i.e.

$$\int_0^T \int_\Omega u \psi_t - \nabla v \cdot \nabla \psi dx dt + \int_\Omega u_0(x) \psi(0, x) dx = 0$$

for all $\varphi \in C^1([0, T] \times \overline{\Omega})$ with $\varphi(T, x) = 0$ for all $x \in \Omega$.

- (4) for all $g \in C^1(\mathbb{R})$ with $g' \geq 0$, define

$$G(x) = \int_0^x g(\phi(\tau)) d\tau \quad \text{and} \quad G^\star(u) = \sum_{i=1}^3 \lambda_i G(S_i(v)).$$

Then the following entropy inequality holds:

$$(1.7) \quad \int_0^T \int_{\Omega} G^*(u) \varphi_t - g(v) \nabla v \nabla \varphi - g'(v) |\nabla v|^2 \varphi dx dt \geq 0$$

for all $\varphi \in \mathcal{D}([0, T] \times \Omega)$ with $\varphi(t, x) \geq 0$ for all $(t, x) \in [0, T] \times \Omega$.

It has been proved [24, 29] that problem 1.1 allows for an entropy Young measure solution, but in general these solutions are not unique, as shown for instance in [33]. Uniqueness of Young measure solutions has been proved by Mascia, Terracina and Tesei [25] under the additional constraint that the initial data and the solution do not take value in the interval (u^-, u^+) . For a detailed discussion of the Young measure solutions to problem 1.1, we refer to [8, 24, 25, 29, 33].

1.2. The Radon measure solution in the case $u^+ = +\infty$. The Neumann initial-boundary value problem for 1.3 under the hypothesis that $u^+ = +\infty$ has been studied by Padron [27]. In analogy to the case where $u^+ < +\infty$, if ϕ is Lipschitz continuous and the initial data is $L^\infty(\Omega)$, then there exists a unique classical solution (u_η, v_η) to 1.3, with $u \in C^1([0, T], L^\infty(\Omega))$ and $v_\eta = \phi(u) + \eta u_t$. However, while the sequence $\{v_\eta\}_{\eta>0}$ is still uniformly bounded in the L^∞ norm, the sequence $\{u_\eta\}_{\eta>0}$ is not, so we cannot take the weak- \star limit of $\{u_\eta\}_{\eta>0}$, even in the sense of Young measures. Nevertheless, thanks to the Neumann boundary conditions, $\|u_\eta(t)\|_1 = \|u_0\|_1$ for all $t \geq 0$. As a consequence, the sequence $\{u_\eta\}_{\eta>0}$ has a limit point u in the space of positive Radon measures over $[0, T] \times \Omega$. In [32], it is proved that u can be represented as the sum $u = \bar{u} + \mu$, where \bar{u} is the baricenter of the Young measure $\nu(t, x)$ associated to an equi-integrable subsequence of $\{u_\eta\}_{\eta>0}$, and μ is a Radon measure over $[0, T] \times \Omega$. As a consequence, the Radon measure u and the L^∞ function v are a weak solution to problem 1.1 in the sense that

$$(1.8) \quad \int_0^T \langle \mu, \varphi_t \rangle_{\mathcal{D}'(\Omega)} dt + \int_0^T \int_{\Omega} \bar{u} \varphi_t - \nabla v \nabla \varphi dx dt + \int_{\Omega} u_0(x) \varphi(0, x) dx = 0,$$

for any $\varphi \in C^1([0, T] \times \bar{\Omega})$ with $\varphi(T, x) = 0$ for all $x \in \Omega$. In particular, a notion of entropy Radon measure solution can be defined for equation 1.1 even in the case $u^+ = +\infty$.

Definition 1.3. An entropy Radon measure solution of problem 1.1 consists of functions $\bar{u}, v, \lambda_i \in L^\infty([0, +\infty) \times \Omega)$, $i = 1, 2$ and of a positive Radon measure $\mu \in \mathbb{M}([0, T] \times \Omega)$, satisfying the conditions:

- (1) $\lambda_i \geq 0$, $\sum_{i=1}^2 \lambda_i = 1$;
- (2) $v \in L^2([0, +\infty), H^1(\Omega))$ and

$$\bar{u} = \begin{cases} \sum_{i=1}^2 \lambda_i S_i(v) & \text{if } v(x) > 0 \\ 0 & \text{if } v(x) = 0; \end{cases}$$

- (3) $(\bar{u} + \mu)_t = \Delta v$ in the sense of equation 1.8;
- (4) the entropy inequality 1.7 holds for \bar{u} and v .

Smarrazzo proved in [32] that problem 1.1 allows for a global entropy Radon measure solution, and we refer to her paper for an in-depth analysis of the properties of such solutions.

We conclude the discussion of the entropy Radon measure solution to problem 1.1 by recalling two features of the singular part of the solution. In [32], Smarrazzo showed that the singular part μ of the entropy Radon measure solution satisfies the following equality for all $t \geq 0$:

$$(1.9) \quad \mu(t) = \left(\int_{\Omega} u_0(x) dx - \int_{\Omega} \bar{u}(t, x) dx \right) \tilde{\mu}(t),$$

where $\tilde{\mu}(t)$ is a positive probability measure over Ω . Moreover, she conjectured that this singular term prevails over the regular term for large times. In section 4, we will show that the solution to problem 1.1 obtained from the grid function formulation satisfies an equality analogous to 1.9, and in section 5 we will show that the conjecture by Smarrazzo holds for such solutions.

1.3. Further remarks on problems 1.1 and 1.2. In [31], Slemrod showed that an approach similar to the one in [29] can be used to obtain Young measure solutions to problem 1.2. Notice however that, depending on the choice of the regularized problem, there are different notions of Young measure solutions for problems 1.1 and 1.2. For instance, Demoulini [6] has given a notion of Young measure solution to problem 1.2 based on a discrete-in-time energy minimization approach that sacrifices some of the physical meaning of the Sobolev approximation in favour of stability of the solution. For a comparison between Demoulini's solutions and the solutions obtained via the Sobolev approximation, see Horstmann and Schweizer [16].

Due to the wide range of their applications to the description of physical phenomena, numerical approximations of equations 1.1 and 1.2 have also been widely studied, especially focusing on particular choices of function ϕ (see for instance [14, 15, 19, 23, 28, 34]). The somewhat surprising feature of problems 1.1 and 1.2 is that, despite their ill-posedness and the absence of solutions in classical functional spaces, they nevertheless allow for stable numerical schemes that lead to successful applications. In particular, in addition to the original Perona-Malik algorithm for edge enhancing [28], the discrete-in-space counterparts of equations 1.1 and 1.2 have been shown to have a well-defined unique solution and their equilibria have been studied in depth, for instance by Lizana and Padron [19] and by Witelski, Shaeffer and Shearer [34].

2. GRID FUNCTIONS AND THEIR PROPERTIES

In this section, we recall some properties of grid functions studied in [4] that will be useful for the grid function formulation of the ill-posed problem 1.1. For an in-depth discussion of the theory of grid functions, we refer to

[4]. We will assume the reader to be familiar with the basics of nonstandard analysis; a classic reference is Davis [5] or Goldblatt [12].

Definition 2.1. Let $N_0 \in {}^*\mathbb{N}$ be an infinite hypernatural number. Set $N = N_0!$ and $\varepsilon = 1/N$, and define

$$\mathbb{X} = \{n\varepsilon : n \in [-N^2, N^2] \cap {}^*\mathbb{Z}\}.$$

We will say that an internal function $f : \mathbb{X}^k \rightarrow {}^*\mathbb{R}$ is a grid function and, if $A \subseteq \mathbb{X}^k$ is internal, we denote by ${}^*\mathbb{R}^A$ the space of grid functions defined over A : ${}^*\mathbb{R}^A = \mathbf{Intl}({}^*\mathbb{R}^A) = \{f : A \rightarrow {}^*\mathbb{R} \text{ and } f \text{ is internal}\}.$

Define also $\Omega_{\mathbb{X}} = {}^*\Omega \cap \mathbb{X}^k$. Notice that $\Omega_{\mathbb{X}}$ is an internal subset of \mathbb{X}^k , and in particular it is hyperfinite. Moreover, Proposition 1.5 of [4] and the hypothesis that Ω is open ensure that ${}^\circ\Omega_{\mathbb{X}} = \overline{\Omega}$. We will say that $x \in \Omega_{\mathbb{X}}$ is nearstandard in Ω iff there exists $y \in \Omega$ such that $x \approx y$.

Definition 2.2 (Grid derivative). For an internal grid function $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, we define the i -th forward finite difference of step ε as

$$\mathbb{D}_i f(x) = \mathbb{D}_i^+ f(x) = \frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon}$$

and the i -th backward finite difference of step ε as

$$\mathbb{D}_i^- f(x) = \frac{f(x) - f(x - \varepsilon e_i)}{\varepsilon}.$$

If $n \in {}^*\mathbb{N}$, \mathbb{D}_i^n is recursively defined as $\mathbb{D}_i(\mathbb{D}_i^{n-1})$ and, if α is a multi-index, then \mathbb{D}^α is defined as expected:

$$\mathbb{D}^\alpha f = \mathbb{D}_1^{\alpha_1} \mathbb{D}_2^{\alpha_2} \dots \mathbb{D}_n^{\alpha_n} f.$$

These definitions can be extended to \mathbb{D}^- by replacing every occurrence of \mathbb{D} with \mathbb{D}^- .

Definition 2.3 (Grid integral and inner product). Let $f, g : {}^*\Omega \rightarrow {}^*\mathbb{R}$ and let $A \subseteq \Omega_{\mathbb{X}} \subseteq \mathbb{X}^k$ be an internal set. We define

$$\int_A f(x) d\mathbb{X}^k = \varepsilon^k \cdot \sum_{x \in A} f(x)$$

and

$$\langle f, g \rangle = \int_{\mathbb{X}^k} f(x) g(x) d\mathbb{X}^k = \varepsilon^k \cdot \sum_{x \in \mathbb{X}^k} f(x) g(x),$$

with the convention that, if $x \notin {}^*\Omega$, $f(x) = g(x) = 0$.

For further details about the properties of the grid derivative and the grid integral, we refer to [20, 2, 3, 4, 13, 18].

We will now recall some well-known facts about S-continuity. This property has been widely used as a bridge between discrete functions of non-standard analysis and standard continuous functions, and it will allow us to define a grid function counterpart of the space of test functions.

Definition 2.4. We say that a function $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ is S -continuous on $\Omega_{\mathbb{X}}$ iff $f(x)$ is finite for some nearstandard $x \in \Omega_{\mathbb{X}}$ and for every nearstandard $x, y \in \Omega_{\mathbb{X}}$, $x \approx y$ implies $f(x) \approx f(y)$.

We also define functions of class S^α for every multi-index α :

- f is of class $S^0(\Omega_{\mathbb{X}})$ if f is S -continuous on $\Omega_{\mathbb{X}}$;
- f is of class $S^\alpha(\Omega_{\mathbb{X}})$ if $\mathbb{D}^\alpha f \in S^0(\Omega_{\mathbb{X}})$ for any standard multi-index α .

Notice that if $f \in S^\alpha(\Omega_{\mathbb{X}})$ for some standard multi-index α , then $f(x)$ is finite at all nearstandard $x \in \Omega_{\mathbb{X}}$.

Definition 2.5 (Algebra of test functions). We define the algebra of test functions over $\Omega_{\mathbb{X}}$ as follows:

$$\mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}}) = \{f \in S^\infty(\Omega_{\mathbb{X}}) : \text{supp } f \subset\subset \Omega\}.$$

In Lemma 2.2 of [4], it is proved that the algebra of test function is the grid function counterpart of the space of standard test functions $\mathcal{D}(\Omega)$, and that if $\varphi \in \mathcal{D}(\Omega)$, then the restriction of ${}^*\varphi$ to $\Omega_{\mathbb{X}}$ belongs to $\mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$.

Let $f, g \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$. We say that $f \equiv g$ iff for all $\varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ it holds $\langle f, \varphi \rangle \approx \langle g, \varphi \rangle$. We will call π the projection from ${}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ to the quotient ${}^*\mathbb{R}^{\Omega_{\mathbb{X}}}/\equiv$, and we will denote by $[f]$ the equivalence class of f with respect to \equiv .

In [4] it is proved that the grid functions can be seen as generalized distributions, and that the finite difference operation generalizes the distributional derivative to the space of grid functions.

Theorem 2.6. Let $\mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) = \{f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \mid \langle f, \varphi \rangle \text{ is finite for all } \varphi \in \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})\}$. The function $\Phi : (\mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv) \rightarrow \mathcal{D}'(\Omega)$ defined by

$$\langle \Phi([f]), \varphi \rangle_{\mathcal{D}'(\Omega)} = {}^\circ \langle f, {}^*\varphi \rangle$$

is an isomorphism of real vector spaces. Moreover, the diagram

$$\begin{array}{ccc} \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) & \xrightarrow{\mathbb{D}^+} & \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) \\ \Phi \circ \pi \downarrow & & \downarrow \Phi \circ \pi \\ \mathcal{D}'(\Omega) & \xrightarrow{D} & \mathcal{D}'(\Omega) \end{array}$$

and the diagram

$$\begin{array}{ccc} \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) & \xrightarrow{\mathbb{D}^-} & \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) \\ \Phi \circ \pi \downarrow & & \downarrow \Phi \circ \pi \\ \mathcal{D}'(\Omega) & \xrightarrow{D} & \mathcal{D}'(\Omega) \end{array}$$

commute.

Proof. See Theorem 2.9 and Theorem 2.15 of [4]. \square

From now on, if $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, we identify the equivalence class $[f]$ with the distribution $\Phi([f])$.

Definition 2.7 (Grid gradient and grid divergence). *If $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, we define the forward and backward grid gradient of f as:*

$$\nabla_{\mathbb{X}}^{\pm} f = (\mathbb{D}_1^{\pm} f, \dots, \mathbb{D}_i^{\pm} f, \dots, \mathbb{D}_k^{\pm} f).$$

In a similar way, if $f : \Omega_{\mathbb{X}} \rightarrow {}^\mathbb{R}^k$, we define the forward and backward grid divergence as*

$$\operatorname{div}_{\mathbb{X}}^{\pm}(f(x, t)) = \sum_{i=1}^k \mathbb{D}_i^{\pm} f(x, t).$$

In the sequel, we will mostly drop the symbol $+$ from the above definitions: for instance, we will write $\nabla_{\mathbb{X}}$ instead of $\nabla_{\mathbb{X}}^+$.

It is a consequence of Theorem 2.6 that, if $f \in S^{(1, \dots, 1)}(\Omega_{\mathbb{X}})$, then ${}^{\circ}(\nabla_{\mathbb{X}}(f))$ is the usual gradient of ${}^{\circ}f$, and similar results holds for $\nabla_{\mathbb{X}}^-$, $\operatorname{div}_{\mathbb{X}}$, $\operatorname{div}_{\mathbb{X}}^-$ and $\Delta_{\mathbb{X}}$. Moreover, by Theorem 2.6, the operators $\nabla_{\mathbb{X}}$ and $\nabla_{\mathbb{X}}^-$ satisfy the formula

$${}^{\circ}\langle \nabla_{\mathbb{X}} f, {}^*\varphi \rangle = {}^{\circ}\langle \nabla_{\mathbb{X}}^- f, {}^*\varphi \rangle = -\langle [f], \operatorname{div} \varphi \rangle_{\mathcal{D}'(\Omega)}$$

for all $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and for all functions $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^k)$, and $\operatorname{div}_{\mathbb{X}}$ and $\operatorname{div}_{\mathbb{X}}^-$ satisfy the formula

$${}^{\circ}\langle \operatorname{div}_{\mathbb{X}} f, {}^*\varphi \rangle = {}^{\circ}\langle \operatorname{div}_{\mathbb{X}}^- f, {}^*\varphi \rangle = -\langle [f], \nabla \varphi \rangle_{\mathcal{D}'(\Omega)}$$

for all $f \in \mathcal{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}, {}^*\mathbb{R}^k)$ and for all $\varphi \in \mathcal{D}(\Omega)$.

Definition 2.8. *For all $f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, define*

$$\|f\|_p^p = \varepsilon^k \sum_{x \in \Omega_{\mathbb{X}}} |f(x)|^p \text{ if } 1 \leq p < \infty, \text{ and } \|f\|_{\infty} = \max_{x \in \Omega_{\mathbb{X}}} |f(x)|.$$

Moreover, $\|f\|_1 = \|f\|_2 + \|\nabla_{\mathbb{X}}^+ f\|_2$.

If $f \in {}^\mathbb{R}^{\Omega_{\mathbb{X}}}$, let $\hat{f} \in {}^*L^2(\mathbb{R}^k)$ defined by*

$$\hat{f}(x) = \begin{cases} f((n_1, n_2, \dots, n_k)\varepsilon) & \text{if } n_i\varepsilon \leq x_i < (n_i + 1)\varepsilon \text{ for all } 1 \leq i \leq k \\ 0 & \text{if } |x_i| > N \text{ for some } 1 \leq i \leq k, \end{cases}$$

with the agreement that $f((n_1, n_2, \dots, n_k)\varepsilon) = 0$ if $(n_1, n_2, \dots, n_k)\varepsilon \notin \Omega_{\mathbb{X}}$.

*Let $P : {}^*L^2(\mathbb{R}^k) \rightarrow {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ be the L^2 projection over ${}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$, i.e. for all $f \in {}^*L^2(\mathbb{R}^k)$, $P(f)$ is the unique grid function satisfying*

$$\langle P(f), g \rangle = {}^*\int_{\Omega} f(x) \hat{g}(x) dx$$

for all $g \in {}^\mathbb{R}^{\Omega_{\mathbb{X}}}$.*

3. THE GRID FUNCTION FORMULATION FOR THE ILL-POSED PDE

In this section, we will derive the grid function formulation for the ill-posed problem 1.1 from very simple basic principles by using an elementary description that generalizes the nonstandard model for the diffusion equation by Hanqiao, St. Mary and Wattenberg [13]. This approach will allow us to choose a suitable grid function counterpart to the operator $\Delta\phi(u)$. Under

the hypotheses 1.1 over ϕ , we will prove that the grid function formulation always has a unique well-defined solution. At the end of this section, we will discuss the coherence of the solution to the grid function formulation with the notions of solutions discussed in section 1.

3.1. Derivation of the grid function formulation. For a matter of commodity, in the derivation of the model we will use the image of a population that moves around the grid \mathbb{X}^k according to some basic rules. The initial distribution of the population around the grid is described by an internal function $u_0 : \mathbb{X}^k \rightarrow {}^*[0, 1]$ satisfying $\int_{\mathbb{X}^k} u_0(x) d\mathbb{X}^k = c \in {}^*\mathbb{R}$. The value $u_0(x)$ determines the number of individuals of the population inhabiting point x at time $t = 0$.

Let $\varepsilon_t = \varepsilon^2$. The population moves around the grid according to the following rules:

- the n -th move occurs between time $(n - 1)\varepsilon_t$ and $n\varepsilon_t$;
- at each jump the population at each grid point breaks into $(2k + 1)$ smaller groups:
 - for $i = 1, \dots, k$, a fraction $p_i(u((n - 1)\varepsilon_t, x))$ of the population at x jumps to $x + \varepsilon \vec{e}_i$;
 - for $i = 1, \dots, k$, a fraction $p_i(u((n - 1)\varepsilon_t, x))$ of the population at x jumps to $x - \varepsilon \vec{e}_i$;
 - the remaining fraction $1 - 2 \sum_{i=1}^k p_i(u((n - 1)\varepsilon_t, x))$ of the population at x remains at x .

In the above description, the functions p_i are internal functions $p_i : {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}$ satisfying

- $0 \leq p_i(r)$ for all $r \in {}^*\mathbb{R}$;
- $\sum_{i=1}^k p_i(r) \leq 1/2$ for all $r \in {}^*\mathbb{R}$

for all $i = 1, \dots, k$. The properties of the functions p_i determine the criteria used by the population to choose whether and how to jump to a nearby grid point. In particular, in the model outlined above an individual chooses its next movement to move according only to local informations. If the functions p_i are constant and do not depend on i , then the above model coincides with the nonstandard model of diffusion discussed in [13]. More complex behaviour can be described by different choices of functions p_i and by introducing a spatial bias.

If we denote by $u(t, x)$ the population present at time t at point x , then by arguing as in section III of [13] we deduce that $u(t, x)$ evolves according

to the finite difference initial value problem

$$\begin{aligned}
u(0, x) &= u_0(x) \\
u((n+1)\varepsilon_t, x) &= \left(1 - 2 \sum_{i=1}^k p_i(u(n\varepsilon_t, x))\right) u(n\varepsilon_t, x) \\
&\quad + \sum_{i=1}^k p_i(u(n\varepsilon_t, x + \varepsilon e_i)) u(n\varepsilon_t, x + \varepsilon e_i) \\
&\quad + \sum_{i=1}^k p_i(u(n\varepsilon_t, x - \varepsilon e_i)) u(n\varepsilon_t, x - \varepsilon e_i)
\end{aligned}$$

From the above equation, if we define $\phi_i(u(n\varepsilon_t, x)) = p_i(n\varepsilon_t, x)u(n\varepsilon_t, x)$, we obtain

$$u((n+1)\varepsilon_t, x) - u(n\varepsilon_t, x) = \sum_{i=1}^k \left[\phi_i(u(n\varepsilon_t, x + \varepsilon e_i)) - 2\phi_i(u(n\varepsilon_t, x)) + \phi_i(u(n\varepsilon_t, x - \varepsilon e_i)) \right].$$

At this point, we divide both sides of the above equation by $\varepsilon_t = \varepsilon^2$ and obtain

$$\frac{u((n+1)\varepsilon_t, x) - u(n\varepsilon_t, x)}{\varepsilon_t} = \sum_{i=1}^k \mathbb{D}_i^+ \mathbb{D}_i^- \phi_i(u(x, t)).$$

If $\phi_i = \phi$ for all $i = 1, \dots, k$, i.e. if the population moves without spatial bias, from the above equality we deduce

$$(3.1) \quad \frac{u((n+1)\varepsilon_t, x) - u(n\varepsilon_t, x)}{\varepsilon_t} = \Delta_{\mathbb{X}} \phi(u).$$

Neumann boundary conditions are imposed in the discrete formulation of the Laplacian in the following way: if $x \in \partial_{\mathbb{X}} \Omega_{\mathbb{X}}$, let

$$I_x^+ = \{i : x + \varepsilon e_i \notin \Omega_{\mathbb{X}}\} \text{ and } I_x^- = \{i : x - \varepsilon e_i \notin \Omega_{\mathbb{X}}\}.$$

The Neumann boundary conditions are equivalent to

$$(3.2) \quad \sum_{i \in I_x^+} \mathbb{D}_i^{+*} \phi(u(x)) = 0 \text{ and } \sum_{i \in I_x^-} \mathbb{D}_i^{-*} \phi(u(x)),$$

for all $x \in \partial_{\mathbb{X}} \Omega_{\mathbb{X}}$, so that the first-order discrete approximation of the Laplacian with Neumann boundary conditions is defined by:

$$\begin{aligned}
\Delta_{\mathbb{X}}^* \phi(u(x)) &= - \sum_{i \in I_x^+} \mathbb{D}_i^{-*} \phi(u(x)) + \sum_{i \in I_x^-} \mathbb{D}_i^{+*} \phi(u(x)) + \\
&\quad + \sum_{i \notin I_x^+ \cup I_x^-} \mathbb{D}_i^+ \mathbb{D}_i^- \phi(u(x)).
\end{aligned}$$

The above argument suggests that the functional $F : L^\infty(\Omega) \cap H^1(\Omega) \rightarrow (C^1(\overline{\Omega}))'$ defined by

$$(3.3) \quad \langle F(u), \varphi \rangle_{C^1(\Omega)} = - \int_{\Omega} \nabla \phi(u) \nabla \varphi dx$$

for all $\varphi \in C^1(\overline{\Omega})$ can be represented in the sense of grid functions by $\Delta_{\mathbb{X}}^* \phi$. We will now prove that $\Delta_{\mathbb{X}}^* \phi$ is indeed coherent with F in the sense of Theorem 4.8 of [4]. Notice how condition (1) of Theorem 4.8 of [4] is replaced by a different coherence condition that depends upon the definition of F .

Proposition 3.1. *Let ϕ be a standard function satisfying hypotheses 1.1, and let F be defined by equation 3.3. Then*

- (1) *if $u \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ satisfies $\|u\|_{\infty} \in {}^*\mathbb{R}_{fin}$ and if u and $\mathbb{D}_i^{\pm} u$ are nearstandard in $L^2(\Omega)$, then $[\Delta_{\mathbb{X}}^* \phi(u)] = F([u]) \in (C^1(\overline{\Omega}))'$;*
- (2) *whenever $u, v \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ are nearstandard in $L^\infty(\Omega) \cap H^1(\Omega)$, if $\|u - v\|_{\infty} \approx 0$ and $\|u - v\|_{H^1} \approx 0$, then $[\Delta_{\mathbb{X}}^* \phi(u)] = [\Delta_{\mathbb{X}}^* \phi(v)]$;*
- (3) *for all $u \in L^\infty(\Omega) \cap H^1(\Omega)$, $[\Delta_{\mathbb{X}}^* \phi(P(*u))] = F(u)$.*

Proof. By the discrete summation by parts formula and by taking into account the Neumann boundary conditions 3.2, for all $\varphi \in S^1(\overline{\Omega_{\mathbb{X}}})$ we have the equality

$$(3.4) \quad \langle \Delta_{\mathbb{X}}^* \phi(u), \varphi \rangle = - \langle \nabla_{\mathbb{X}}^- \phi(u(x + \varepsilon)), \nabla_{\mathbb{X}}^+ \varphi \rangle.$$

We will now show that $[\nabla_{\mathbb{X}}^- \phi(u)] = [\phi'(u)] \nabla[u]$. For all $1 \leq i \leq k$, we have the equality

$$\begin{aligned} \mathbb{D}_i^- \phi(u(x)) &= \frac{{}^*\phi(u(x)) - {}^*\phi(u(x - \varepsilon e_i))}{\varepsilon} \\ &= \frac{{}^*\phi(u(x)) - \varphi(u(x) - \varepsilon \mathbb{D}_i^+ u(x - \varepsilon e_i))}{\varepsilon \mathbb{D}_i^+ u(x - \varepsilon e_i)} \cdot \mathbb{D}_i^+ u(x - \varepsilon e_i). \end{aligned}$$

The hypothesis that $\mathbb{D}_i^+ u$ is nearstandard in $L^2(\Omega)$ ensures that there is a L_N -nullset $\Omega_0 \subset \Omega_{\mathbb{X}}$ such that $\mathbb{D}_i^+ u(x - \varepsilon e_i)$ is finite for all $x \in \Omega_{\mathbb{X}} \setminus \Omega_0$. Moreover, if $x \in \Omega_0$, $\varepsilon \mathbb{D}_i^+ u(x - \varepsilon e_i) \approx 0$, otherwise $\mathbb{D}_i^+ u$ would not be nearstandard in $L^2(\Omega)$. As a consequence, $[\mathbb{D}_i^+ u(x - \varepsilon e_i)] = D_i[u](^{\circ}x)$ and, since $\|u\|_{\infty}$ is finite, $[\mathbb{D}_i^- \phi(u)] = [\phi'(u)] D_i^+[u]$. Taking into account these equalities, we deduce

$$\begin{aligned} {}^{\circ} \langle \Delta_{\mathbb{X}}^* \phi(u), \varphi \rangle &= \langle [\Delta_{\mathbb{X}}^* \phi(u)], {}^{\circ} \varphi \rangle_{C^1(\Omega)} \\ &= - \int_{\Omega} [\nabla_{\mathbb{X}}^- \phi(u)] \nabla^{\circ} \varphi dx \\ (3.5) \quad &= - \int_{\Omega} [\phi'(u)] \nabla[u] \nabla^{\circ} \varphi dx \end{aligned}$$

and, by the hypothesis that $\mathbb{D}_i^+ u$ are nearstandard in $H^1(\Omega)$ for $1 \leq i \leq k$, $[u] \in H^1(\Omega)$. As a consequence, the integral 3.5 is finite, and $[\Delta_{\mathbb{X}}^* \phi(u)] \in (C^1(\overline{\Omega}))'$.

In order to prove that $[\Delta_{\mathbb{X}}^* \phi(u)] = F([u])$, notice that the hypothesis that $\|u\|_{\infty} \in {}^*\mathbb{R}_{fin}$ and that u is nearstandard in $L^2(\Omega)$ entail that the Young measure associated to u is Dirac, hence a.e. equal to $[u]$ by Theorem 3.13 of [4]. As a consequence, $[^*\varphi(u)] = \varphi([u])$, so that from equality 3.5 we deduce that $[\Delta_{\mathbb{X}}^* \phi(u)] = F([u])$.

We will now prove that $\|u - v\|_{\infty} \approx 0$ and $\|u - v\|_{H^1} \approx 0$ imply $[\Delta_{\mathbb{X}}^* \phi(u)] = [\Delta_{\mathbb{X}}^* \phi(v)]$. From the hypothesis $\|u - v\|_{\infty} \approx 0$ and by S-continuity of $^*\phi$ and of $^*\phi'$, we have

$$\|{}^*\phi(u) - {}^*\phi(v)\|_{\infty} \approx \|{}^*\phi'(u) - {}^*\phi'(v)\|_{\infty} \approx 0.$$

The assumption $\|u - v\|_{H^1} \approx 0$ entails also $[\nabla_{\mathbb{X}}^+ u] = [\nabla_{\mathbb{X}}^+ v]$, so that

$$[\nabla_{\mathbb{X}}^+ {}^*\phi(u)] = [\phi'(u)] \nabla[u] = [\phi'(v)] \nabla[v] = [\nabla_{\mathbb{X}}^+ {}^*\phi(v)].$$

As a consequence, from equality 3.5 we obtain

$$\langle [\Delta_{\mathbb{X}}^* \phi(u)] - [\Delta_{\mathbb{X}}^* \phi(v)], {}^{\circ}\varphi \rangle_{C^1(\Omega)} = \int_{\Omega} ([\phi'(u)] \nabla^- [u] - [\phi'(v)] \nabla^- [v]) \nabla^{\circ} \varphi dx = 0,$$

so that the proof of (2) is concluded.

Part (3) of the assertion is a consequence of part (1), since if $u \in L^{\infty}(\Omega) \cap H^1(\Omega)$, then Lemma 3.7 of [4] ensures that $P(^*u)$ satisfies the hypotheses of part (1) of the assertion and that $[P(^*u)] = u$. \square

3.2. The grid function formulation for the ill-posed PDE. We now have all of the elements to formulate problem 1.1 in the sense of grid functions.

Definition 3.2. *The functions $[u], [^*\phi(u)] \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}/ \equiv$ are called a grid solution of 1.1 if u satisfies the following system of ODEs:*

$$(3.6) \quad \begin{cases} u_t = \Delta_{\mathbb{X}}^* \phi(u); \\ u(0, x) = {}^*u_0(x). \end{cases}$$

Remark 3.3. *A standard version of the formulation 3.6 with $\Omega_{\mathbb{X}} = [0, 1]$ and with standard N has been used by Lizana and Padron [19] to describe the dynamics of a population inhabiting a finite collection of $N+1$ equally spaced points $\{0, \dots, i/N, \dots, 1\}$ on the interval $[0, 1]$. By the transfer principle, many properties of the finite model discussed in section 3 of [19] hold also for the hyperfinite system 3.6. Conversely, many of the results discussed in Sections 5 and 6 of this paper can be applied to this finite model by omitting the stars and by taking $N \in \mathbb{N}$.*

Remark 3.4. *Notice that problem 3.6 makes sense for an arbitrary $f : {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}$ instead of $^*\phi$ and for arbitrary initial data. However, since we are interested not only in the solutions to problem 3.6, but also in the coherence with the measure-valued solutions to problem 1.1, we will restrict our attention to the case where $f = ^*\phi$, and ϕ satisfies hypotheses 1.1, and where the initial data is the nonstandard extension of a function $u_0 \in L^{\infty}(\Omega)$ that satisfies $u_0(x) \geq 0$ for all $x \in \Omega$.*

Problem 3.6 can be interpreted as a hyperfinite system of ordinary differential equations: as such, the existence of solutions and their properties can be studied by the theory of ordinary differential equations. These results, in turn, apply to the grid solution for problem 1.1.

Theorem 3.5. *There exists a maximal interval $I \subseteq {}^*\mathbb{R}$ such that Problem 3.6 has a unique solution $u \in {}^*C^1(I, {}^*\mathbb{R}^{\Omega_{\mathbb{X}}})$; Moreover, $\|u(t)\|_1 = \|u_0\|_1$ for all $t \in I$.*

Proof. By transfer, existence and uniqueness can be deduced from the theory of ordinary differential equations.

In order to prove that $\|u(t)\|_1 = \|u_0\|_1$ for all $t \in I$, notice that it holds

$$\frac{d}{dt} \int_{\Omega_{\mathbb{X}}} u(t, x) d\mathbb{X}^k = \int_{\Omega_{\mathbb{X}}} u_t(t, x) d\mathbb{X}^k = \int_{\Omega_{\mathbb{X}}} \Delta_{\mathbb{X}} \phi(u(t, x)) d\mathbb{X}^k.$$

Thanks to the Neumann boundary conditions, $\int_{\Omega_{\mathbb{X}}} \Delta_{\mathbb{X}} \phi(u(t, x)) d\mathbb{X}^k = 0$, so that the mass of the solution is preserved. \square

Proposition 3.6 (Invariant set). *For all $t \in I$ and for all $x \in \Omega_{\mathbb{X}}$,*

- (1) *if $u^+ = +\infty$, then $u(t, x) \geq 0$.*
- (2) *if $u^+ < +\infty$, then $u(t, x) \in \{0, \max\{\|{}^*u_0\|_{\infty}, S_3(\phi(u^-))\}\}$. In particular, $\|u(t)\|_{\infty} \in {}^*\mathbb{R}_{fin}$ is homogeneously bounded for all $t \geq 0$.*

Proof. If $u^+ = +\infty$, let

$$\bar{t} = \sup\{t \geq 0 : u(t, x) \geq 0 \text{ for all } t \in I \text{ and } x \in \Omega_{\mathbb{X}}\}.$$

The hypotheses over ϕ and the definition of \bar{t} ensure that if $u(\bar{t}, x) = 0$, then $u_t(\bar{t}, x) = \Delta_{\mathbb{X}} \phi(u(\bar{t}, x)) \geq 0$. As a consequence, $\bar{t} = \sup I$.

Similarly, if $u^+ < +\infty$, let

$$\bar{t} = \sup\{t \geq 0 : u(t, x) \in \{0, \max\{\|{}^*u_0\|_{\infty}, S_3(\phi(u^-))\}\} \text{ for all } t \in I \text{ and } x \in \Omega_{\mathbb{X}}\}.$$

In this case, if $u(\bar{t}, x) = 0$, the equality $u_t(\bar{t}, x) = \Delta_{\mathbb{X}} \phi(u(\bar{t}, x)) \geq 0$ holds as in the previous part of the proof. If $u(\bar{t}, x) = \max\{\|{}^*u_0\|_{\infty}, \phi(u^-)\}$, then a similar calculation allows to conclude $u_t(\bar{t}, x) \leq 0$. We deduce that it holds $\bar{t} = \sup I$ also for this case. \square

Since for any initial data $u_0 \in L^{\infty}(\Omega)$ the invariant set for the dynamical system 3.6 is bounded, we deduce global existence in time.

Corollary 3.7 (Global existence in time). *The solution u of system 3.6 satisfies $u \in {}^*C^1({}^*[0, \infty), {}^*\mathbb{R}^{\Omega_{\mathbb{X}}})$.*

Proof. Let u be the solution of Problem 3.6, and let I be the interval over which u is defined. Define also

$$\mathbb{S}^+({}^*u_0) = \{f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} : f(x) \geq 0 \text{ for all } x \in \Omega_{\mathbb{X}} \text{ and } \|f\|_1 = \|{}^*u_0\|_1\}.$$

By Theorem 3.5 and by Proposition 3.6, $u(t) \in \mathbb{S}^+({}^*u_0)$ for all $t \in I$.

Let $\Omega_{\mathbb{X}} = \{x_1, \dots, x_M\}$, with $M = |\Omega_{\mathbb{X}}|$. We identify u with a vector-valued function that, by abuse of notation, we will still denote by $u : I \rightarrow$

${}^*\mathbb{R}^M$, with the convention that the k -th component of $u(t)$ is $u(t, x_k)$. Since the set $\mathbb{S}^+({}^*u_0)$ is * compact in ${}^*\mathbb{R}^M$, the theory of ODEs allows to conclude that u has global existence in time. \square

As a consequence of Theorem 3.5 and of Corollary 3.7, we deduce that problem 1.1 always has a unique global grid solution.

4. COHERENCE OF THE GRID SOLUTION WITH THE MEASURE-VALUED SOLUTIONS TO THE ILL-POSED PDE

This section is devoted to the study of the coherence of the grid solution with the notions of measure-valued solutions for problem 1.1 discussed in section 1. In particular, we will show that, if u is regular enough, then the grid solution of problem 1.1 coincides with an entropy Young measure solution in the case where $u^+ < +\infty$, and with the entropy Radon measure solution in the case where $u^+ = +\infty$.

Our argument relies on an equality that will be used to establish an entropy condition for the pair $[u], [{}^*\phi(u)]$.

Lemma 4.1. *For all internal $f, g : \mathbb{X}^k \rightarrow {}^*\mathbb{R}$, it holds*

$$\operatorname{div}_{\mathbb{X}}^-(g(f(x)) \cdot \nabla_{\mathbb{X}} f(x)) = g(f(x)) \Delta_{\mathbb{X}}(f(x)) + \nabla_{\mathbb{X}}^- f(x) \cdot \nabla_{\mathbb{X}}^- g(f(x)).$$

Notice that the above result is independent of the regularity of f and g .

Lemma 4.2 (Entropy condition). *For any $g \in C^1(\mathbb{R})$ with $g' \geq 0$, define $G(u(t, x)) = \int_0^{u(t, x)} g(\phi(s)) ds$. Then, if u is the solution to problem 3.6, it holds*

$${}^*G(u)_t = \operatorname{div}_{\mathbb{X}}^-(({}^*g(\phi(u)) \nabla_{\mathbb{X}}^+(\phi(u))) - \nabla_{\mathbb{X}}^- {}^*g'(\phi(u)) \cdot \nabla_{\mathbb{X}}^- \phi(u).$$

and, if $\nabla_{\mathbb{X}}^- \phi(u)$ is finite,

$$(4.1) \quad {}^*G(u)_t \approx \operatorname{div}_{\mathbb{X}}^-(({}^*g(\phi(u)) \nabla_{\mathbb{X}}^+(\phi(u))) - {}^*g'(\phi(u)) |\nabla_{\mathbb{X}}^- \phi(u)|^2).$$

Proof. For G , g and u it holds

$$G(u)_t = g(\phi(u)) u_t = g(\phi(u)) \Delta_{\mathbb{X}} \phi(u).$$

By Lemma 4.1,

$$\operatorname{div}_{\mathbb{X}}^-((g(\phi(u)) \nabla_{\mathbb{X}}^+(\phi(u))) = g(\phi(u)) \Delta_{\mathbb{X}} \phi(u) + \nabla_{\mathbb{X}}^- g(\phi(u)) \cdot \nabla_{\mathbb{X}}^- \phi(u),$$

so that the first equality is proved.

For the second equality, we have already shown in the proof of Proposition 3.1 that if $\nabla_{\mathbb{X}}^- \phi(u)$ is finite, then $\nabla_{\mathbb{X}}^- g(\phi(x)) \approx g'(\phi(x)) \nabla_{\mathbb{X}}^- \phi(x)$. As a consequence,

$$\nabla_{\mathbb{X}}^- g(\phi(x)) \cdot \nabla_{\mathbb{X}}^- \phi(x) \approx g'(\phi(x)) |\nabla_{\mathbb{X}}^- \phi(x)|^2,$$

as desired. \square

Formula 4.1 can be regarded as an entropy condition for system 3.6. In particular, this equality allows us to prove that the solution obtained by the nonstandard model 3.6 retains the physical meaning of an entropy solution.

Now, we will prove that the grid solution of problem 1.1 is always a very weak solution in the sense of distributions.

Lemma 4.3. *Let u be the solution of problem 3.6. Then $[u], [* \phi(u)] \in \mathcal{D}'(\mathbb{R} \times \Omega)$ is a very weak solution of problem 1.1 in the sense of distributions, i.e. $[u]$ and $[* \phi(u)]$ satisfy*

$$(4.2) \quad \int_0^T \langle [u], \varphi_t \rangle + \langle [* \phi(u)], \Delta \varphi \rangle dt + \int_{\Omega} u_0(x) \varphi(0, x) dx = 0$$

for all $\varphi \in C^1([0, T], \mathcal{D}'(\Omega))$ with $\varphi(T, x) = 0$ for all $x \in \Omega$.

Proof. By Proposition 3.6, $\|u(t)\|_1 = c \in {}^*\mathbb{R}_{fin}$ for all $t \in {}^*\mathbb{R}_{\geq 0}$ and, by Proposition 3.6 if $u^+ < +\infty$ or by the boundedness of ϕ if $u^+ = +\infty$, also $\|* \phi(u)\|_{\infty} \in {}^*\mathbb{R}_{fin}$ for all $t \in {}^*\mathbb{R}_{\geq 0}$.

Now let $\varphi \in C^1([0, T], \mathcal{D}'(\Omega))$ with $\varphi(T, x) = 0$ for all $x \in \Omega$, and define $\varphi_{\mathbb{X}}(t) = * \varphi(t)|_{\mathbb{X}}$. Since $u \in {}^*C^1({}^*\mathbb{R}_{\geq 0}, {}^*\mathbb{R}^{\Omega_{\mathbb{X}}})$ and $\varphi_{\mathbb{X}} \in {}^*C^1([0, T], \mathcal{D}_{\mathbb{X}}(\Omega_{\mathbb{X}}))$, we have

$$(4.3) \quad \int_0^T \langle u_t(t), \varphi_{\mathbb{X}}(t) \rangle dt = - \int_0^T \langle u(t), (\varphi_{\mathbb{X}})_t(t) \rangle dt - \langle {}^*u_0, \varphi_{\mathbb{X}}(0, x) \rangle.$$

By the discrete summation by parts formula, for all $t \in {}^*\mathbb{R}_{\geq 0}$ we have

$$(4.4) \quad \langle \Delta_{\mathbb{X}} \phi(u(t)), \varphi_{\mathbb{X}}(t) \rangle = \langle \phi(u(t)), \Delta_{\mathbb{X}} \varphi_{\mathbb{X}}(t) \rangle.$$

Taking into account that u satisfies 3.6, by equations 4.3 and 4.4, we obtain

$$\int_0^T \langle u(t), (\varphi_{\mathbb{X}})_t(t) \rangle + \langle \phi(u(t)), \Delta_{\mathbb{X}} \varphi_{\mathbb{X}}(t) \rangle dt + \langle {}^*u_0, \varphi_{\mathbb{X}}(0, x) \rangle = 0.$$

By Lemma 3.6 of [4],

$$\circ \langle {}^*u_0, \varphi_{\mathbb{X}}(0, x) \rangle = \int_{\Omega} u_0 \varphi(0, x) dx.$$

As a consequence, $[u]$ and $[* \phi(u)]$ satisfy

$$\circ \left(\int_0^T \langle u(t), {}^* \varphi(t) \rangle dt \right) = \int_{[0, T]} \langle [u], \varphi_t \rangle_{\mathcal{D}'(\Omega)} dt + \int_{\Omega} u_0(x) \varphi(0, x) dx$$

and

$$\circ \left(\int_0^T \langle \Delta_{\mathbb{X}} \phi(u(t)), {}^* \varphi(t) \rangle dt \right) = \int_0^T \langle [* \phi(u)], \Delta \varphi \rangle_{\mathcal{D}'(\Omega)} dt.$$

As a consequence, we deduce that equality 4.2 holds. \square

4.1. **The case $u^+ < +\infty$.** We will now discuss coherence of the grid solution with the solutions of problem 1.1 in the case where $u^+ < \infty$. As expected, if u is regular enough, then the grid solution to problem 1.1 is a solution of problem 1.1 in a classical sense. The degree of regularity of the standard solution depends upon the regularity of u .

Theorem 4.4. *Let $[u], [* \phi(u)]$ be the grid solution of Problem 3.6, and let $\nu(t, x)$ the Young measure associated to u .*

- (1) *If $[* \phi(u)] \in L^2([0, T], H^1(\Omega))$, then $[u], [* \phi(u)]$ is an entropy Young measure solution of Problem 1.1 in the sense of equation 1.6.*
- (2) *Under the hypotheses*
 - $\nu(t, x)$ is Dirac a.e.,
 - $[* \phi(u)] \in L^\infty([0, T] \times \Omega)$,*then $[u] \in L^\infty([0, T], L^\infty(\Omega))$ and $[u], [* \phi(u)]$ is a very weak solution of Problem 1.1.*
- (3) *Under the hypotheses*
 - $\nu(t, x)$ is Dirac a.e.,
 - $[* \phi(u)] \in L^\infty([0, T] \times \Omega) \cap L^2([0, T], H^1(\Omega))$,*then $[u], [* \phi(u)]$ is a weak solution of Problem 1.1.*
- (4) *If $u \in S^1(*[0, +\infty), S^2(\Omega_{\mathbb{X}}))$, then $[u] = {}^\circ u$ is a classical global solution of Problem 1.1.*

Proof. (1). Since $[* \phi(u(t))] \in H^1(\Omega)$ for a.e. $t \geq 0$, we deduce that $\int_\Omega \phi(\tau) d\nu(t, x)$ is single-valued for a.e. $t \geq 0$. In particular, $\nu(t, x)$ is a.e. a superposition of at most three Dirac measures centred at $S_i(\int_\Omega \phi(\tau) d\nu(t, x))$, and $[u]$ is the barycentre of ν in the sense that

$$[u](t, x) = \int_{\mathbb{R}} \tau d\nu(t, x).$$

From these properties, we recover conditions (1)–(3) of the definition of entropy Young measure solution.

By taking into account that $[* \phi(u)] \in H^1(\Omega)$, from Proposition 3.1 and from equation 4.2 we deduce that $[u]$ and $[* \phi(u)]$ satisfy

$$\int_0^T \int_\Omega [u] \varphi_t - \nabla[* \phi(u)] \nabla \varphi dx dt + \int_\Omega u_0(x) \varphi(0, x) dx = 0$$

for all $\varphi \in C^1([0, T] \times \overline{\Omega})$ with $\varphi(T, x) = 0$ for all $x \in \Omega$.

We will now derive the entropy condition 1.7 for $[u]$ and $[* \phi(u)]$. Let $g \in C^1(\mathbb{R})$, $g' \geq 0$, $G(x) = \int_0^x g(\phi(\tau)) d\tau$, and let $\varphi \in \mathcal{D}([0, T] \times \Omega)$ with $\varphi \geq 0$. Define also

$$G^\star([u]) = \sum_{i=1}^3 \int_0^{S_i([* \phi(u)])} g(\tau) d\tau.$$

By Theorem 3.12 of [4], we have the following equalities

$$\begin{aligned}
-\int_0^T \int_{\Omega} G^*([u]) \varphi_t dx dt &= -\circ \int_0^T \langle {}^*G(u), {}^*\varphi_t \rangle dt \\
&= \circ \int_0^T \langle {}^*G(u)_t, {}^*\varphi \rangle dt \\
&= \circ \int_0^T \langle {}^*g({}^*\phi(u))u_t, {}^*\varphi \rangle dt \\
&= \circ \int_0^T \langle {}^*g({}^*\phi(u))\Delta_{\mathbb{X}}\varphi(u), {}^*\psi \rangle dt.
\end{aligned}$$

By Lemma 4.2 and by $[{}^*\phi(u)] \in H^1(\Omega)$, we deduce

$$\int_0^T \langle {}^*g({}^*\phi(u))\Delta_{\mathbb{X}}\varphi, {}^*\varphi \rangle dt \approx \int_0^T \langle \operatorname{div}_{\mathbb{X}}^-({}^*g({}^*\phi(u))\nabla_{\mathbb{X}}^+\phi(u)) - {}^*g'({}^*\phi(u))|\nabla_{\mathbb{X}}^-\phi(u)|^2, {}^*\varphi \rangle dt$$

By the discrete summation by parts formula and by Theorem 2.6,

$$\begin{aligned}
-\int_0^T \langle \operatorname{div}_{\mathbb{X}}^-({}^*g({}^*\phi(u))\nabla_{\mathbb{X}}^+\phi(u)), {}^*\varphi \rangle dt &\approx \int_0^T \langle {}^*g({}^*\phi(u))\nabla_{\mathbb{X}}^+\phi(u), \nabla_{\mathbb{X}}^-{}^*\varphi \rangle dt \\
&\approx \int_0^T \int_{\Omega} g([{}^*\phi(u)]) \nabla[{}^*\phi(u)] \nabla \varphi dx dt
\end{aligned}$$

and, by Proposition 3.3 of [4],

$$\circ \int_0^T \langle {}^*g'({}^*\phi(u))|\nabla_{\mathbb{X}}^-{}^*\phi(u)|^2, {}^*\varphi \rangle dt \geq \int_0^T \int_{\Omega} g'([{}^*\phi(u)]) |\nabla[{}^*\phi(u)]|^2 \varphi dx dt.$$

Putting together the above inequalities, we deduce

$$\int_0^T \int_{\Omega} G^*([u]) \varphi_t - g([{}^*\phi(u)]) \nabla([{}^*\phi(u)]) \nabla \varphi - g'([{}^*\phi(u)]) |\nabla[{}^*\phi(u)]|^2 \varphi dx dt \geq 0,$$

so that $[u]$ and $[{}^*\phi(u)]$ satisfy the entropy condition 1.7.

(2). Since $\nu(t, x)$ is Dirac, by Theorem 3.13 of [4], it coincides with $[u]$ and, as a consequence, we also have $[{}^*\phi(u(t))] = \phi([u])$. By substituting $[u]$ and $\phi([u])$ in equation 4.2, we obtain

$$\int_0^T \int_{\Omega} [u](t, x) \varphi_t + \phi([u])(t, x) \Delta \varphi d(t, x) + \int_{\Omega} u_0(x) \varphi(0, x) dx = 0,$$

that is, $[u]$ and $[{}^*\phi(u)]$ are a very weak solution of Problem 1.1.

(3). In addition to the conclusions of point (2), we also have $\phi([u]) \in L^\infty((0, T), H^1(\Omega))$, so that by Proposition 3.1 the following equality

$$\langle \Delta_{\mathbb{X}}\phi(u(t))(x), {}^*\varphi(t) \rangle = - \int_{\Omega} \nabla \phi([u])(t) \nabla \varphi(t) dx$$

holds for a.e. $t \geq 0$. Hence, by substituting $[u]$ and $[{}^*\phi(u)]$ in equation 4.2, we deduce that $[u]$ and $[{}^*\phi(u)]$ are a weak solution of Problem 1.1.

We will now prove (4). By Theorem 1.14 of [4], $[u] = \circ u$ and, by Corollary 1.16 of [4], $\circ u \in C^1(\mathbb{R}_{\geq 0}, C^2(\Omega))$. Moreover, $\circ^* \phi(u) = \phi(\circ u) \in$

$C^1([0, +\infty), C^2(\Omega))$ and, by Theorem 1.15 of [4], $[\Delta_{\mathbb{X}}^* \phi(u(t))] = \Delta \phi({}^\circ u(t))$ for all $t \geq 0$. The boundary conditions 3.2 ensure that

$$\frac{\partial \phi({}^\circ u)}{\partial \nu} = 0 \text{ in } [0, +\infty) \times \partial \Omega.$$

This is sufficient to conclude that ${}^\circ u$ is a classic global solution of Problem 1.1. \square

4.2. The case $u^+ = +\infty$. We will now discuss coherence of the grid solution to problem 3.6 with the measure-valued solution to equation 1.1 under the hypothesis that $u^+ = +\infty$.

Theorem 4.5. *Let $[u], [* \phi(u)]$ be the grid solution of Problem 3.6, and let $\nu(t, x)$ the Young measure associated to u . If $[* \phi(u)] \in L^2([0, +\infty), H^1(\Omega))$, then $[u], [* \phi(u)]$ is an entropy Radon measure solution of problem 1.1 in the sense of equation 1.8.*

Proof. Let

$$u_r(t, x) = \int_{\mathbb{R}} \tau d\nu(t, x)$$

be the barycentre of ν , and let $\mu(t) = [u](t) - \bar{u}(t)$. The Young measure $\nu(t, x)$ corresponds to the regular term of the solution to problem 1.1, and the Radon measure μ corresponds to the singular term.

The hypothesis $[* \phi(u)](t) \in H^1(\Omega)$ ensures that $[* \phi(u)](t, x)$ is single-valued for a.e. $t \geq 0$ and $x \in \Omega$. If $[* \phi(u)](t, x) = c \neq 0$, this implies that $\nu(t, x)$ is a superposition of at most two Dirac measures centred at $S_1(c)$ and at $S_2(c)$. If $[* \phi(u)](t, x) = 0$, then $\nu(t, x)$ is a Dirac Young measure centred at 0.

Notice that, for any $\varphi \in C^1([0, T] \times \bar{\Omega})$ with $\varphi(T, x) = 0$ for all $x \in \Omega$, we have the equality

$$\int_0^T \langle u, {}^* \varphi \rangle dt = \int_0^T \int_{\Omega} [u] \varphi dx dt = \int_0^T \int_{\Omega} u_r \varphi dx dt + \int_0^T \langle \mu, \varphi \rangle_{\mathcal{D}'(\Omega)} dt,$$

for any arbitrary $T > 0$. By Proposition 3.1, by equality 4.2 and by the hypothesis that $[* \phi(u)](t) \in H^1(\Omega)$, we deduce that u_r , μ and $[* \phi(u)]$ satisfy the equality

$$\int_0^T \langle \mu, \varphi_t \rangle_{\mathcal{D}'(\Omega)} dt + \int_0^T \int_{\Omega} u_r \varphi_t - \nabla[* \phi(u)] \nabla \varphi dx dt + \int_{\Omega} u_0(x) \varphi(0, x) dx = 0,$$

so that $[u]$ and $[* \phi(u)]$ induce a Radon entropy solution to problem 1.1 in the sense of equation 1.8.

The entropy condition under the hypothesis that G is equi-integrable can be deduced from Lemma 4.2 and from an argument analogous to the one in the proof of point (1) of Theorem 4.4. \square

We can also prove that the singular part of the Radon measure solution can be disintegrated as in equation 1.9.

Proposition 4.6. *Let μ be defined as in the proof of Theorem 4.5. There exists a function $\tilde{\mu} : L^\infty([0, +\infty), \mathbb{M}^\mathbb{P}(\Omega))$ such that*

$$\mu(t) = \left(\int_{\Omega} u_0(x) dx - \int_{\Omega} u_r(t, x) dx \right) \tilde{\mu}(t).$$

Moreover, the support of $\tilde{\mu}(t)$ is a null-set with respect to the k -th dimensional Lebesgue measure for all $t \geq 0$.

Proof. By definition of μ , μ can be interpreted as a function $\mu : L^\infty([0, +\infty), \mathbb{M}(\Omega))$ that satisfies

$$\int_{\Omega} u_r(t, x) dx + \int_{\Omega} d\mu(t) = {}^\circ \|u(x, t)\|_1.$$

By Theorem 3.5, ${}^\circ \|u(x, t)\|_1 = \int_{\Omega} u_0(x) dx$, so the first part of the assertion is proved. The second part of the assertion is a consequence of $\|u(t)\|_1 \in {}^*\mathbb{R}_{fin}$. \square

5. ASYMPTOTIC BEHAVIOUR OF THE GRID SOLUTIONS TO THE ILL-POSED PDE

In this section, we will draw conclusions about the asymptotic behaviour of the grid solutions to problem 1.1 by studying the asymptotic behaviour of the solutions to the grid function formulation 3.6. In particular, we will carry out this study by determining the stability of the steady states of problem 3.6.

A steady state of problem 3.6 is a grid function $\tilde{u} \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ that satisfies $\Delta_{\mathbb{X}}^* \phi(\tilde{u}) = 0$. By definition of $\Delta_{\mathbb{X}}^* \phi$, \tilde{u} is a steady state if and only if ${}^* \phi(\tilde{u}(x)) = c$ for all $x \in \Omega_{\mathbb{X}}$. In particular, \tilde{u} can assume up to three values $\omega_1 \in (0, u^-)$, $\omega_2 \in (u^-, u^+)$ and, when $u^+ < +\infty$, $\omega_3 \in (u^+, +\infty)$ satisfying $\phi(\omega_1) = \phi(\omega_2) = \phi(\omega_3)$. By Proposition 3.1, the steady states of the grid function formulation 3.6 induce a steady state for problem 1.1.

Notice however that a steady state of problem 1.1 corresponds to a grid function \tilde{v} that satisfies only the weaker condition $\Delta_{\mathbb{X}}^* \phi(\tilde{v}) \approx 0$. If $\|\tilde{v}\|_\infty \in {}^*\mathbb{R}_{fin}$, then we expect that there exists a steady state \tilde{u} of problem 3.6 with $\|\tilde{u} - \tilde{v}\|_\infty \approx 0$. In this case, the stability of \tilde{v} can be determined by studying the stability of \tilde{u} . On the other hand, if $u^+ = +\infty$ and $\|\tilde{v}\|_\infty \notin {}^*\mathbb{R}_{fin}$, then \tilde{v} induces a measure-valued steady state of problem 1.1, but there might not exist a steady state \tilde{u} of the grid function formulation 3.6 which satisfies $\|\tilde{u} - \tilde{v}\|_\infty \approx 0$. Nevertheless, in section 5.4, we will show that the asymptotic behaviour of the grid solutions to problem 1.1 can be characterized a posteriori from the asymptotic behaviour of the solutions to problem 3.6.

5.1. Asymptotic behaviour of the solutions to problem 3.6. Since problem 3.6 corresponds to a hyperfinite dynamical system, we need to introduce an appropriate notion of stability for its steady states. Our choice is to use the nonstandard counterpart of the classical notion of stability in the L^∞ norm for discrete dynamical systems. In the following definition, it

is useful to keep in mind that $u \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ can be identified with a vector in the euclidean space ${}^*\mathbb{R}^{|\Omega_{\mathbb{X}}|}$.

Definition 5.1. Let $f : {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} \rightarrow {}^*\mathbb{R}$ and let $v(t) : {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ be the solution of the nonstandard differential equation $u' = f(u)$ with initial data $v(0)$. We will say that $u \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}}$ is

- **stable* iff for all $\eta \in {}^*\mathbb{R}$, $\eta > 0$ there exists $\delta \in {}^*\mathbb{R}$, $\delta > 0$ such that $\|u - v(0)\|_{\infty} < \delta$ implies $\|u - v(t)\|_{\infty} < \eta$ for all $t \in {}^*\mathbb{R}_+$;
- **attractive* iff there exists $\rho \in {}^*\mathbb{R}$, $\rho > 0$ such that $\|u - v(0)\|_{\infty} < \rho$ implies ${}^*\lim_{t \rightarrow +\infty} \|u - v(t)\|_{\infty} = 0$;
- *asymptotically *stable* iff it is **stable* and **attractive*;
- *globally asymptotically *stable* iff it is **stable* and for all $v(0) \in \Omega_{\mathbb{X}}$ ${}^*\lim_{t \rightarrow +\infty} |u - v(t)| = 0$;
- **unstable* iff it is not **stable*.

Notice that a necessary condition for u to be **stable* or **attractive* is that $f(u) = 0$, i.e. u must be an equilibrium point of the differential equation.

Since the ${}^*L^{\infty}$ norm over $\Omega_{\mathbb{X}}$ is equivalent to the euclidean norm in ${}^*\mathbb{R}^{|\Omega_{\mathbb{X}}|}$, the stability in the ${}^*L^{\infty}$ norm for the grid function formulation 3.6 can be studied by exploiting the theory of finite dynamical systems.

For the following analysis of the asymptotic behaviour of solutions of system 3.6, we assume that they are isolated in $\mathbb{S}^+({}^*u_0)$, i.e. that there is only a hyperfinite number of steady states in $\mathbb{S}^+({}^*u_0)$. For a discussion of this hypothesis and for sufficient conditions that ensure the existence of a hyperfinite number of steady states in $\mathbb{S}^+({}^*u_0)$, we refer to Lizana and Padron [19]. Their hypothesis is a sharpening of the condition that S'_1, S'_2 and S'_3 must be linearly independent on the spinoidal interval (u^-, u^+) , already discussed in [26].

Proposition 5.2. *If the steady states of 3.6 are isolated in $\mathbb{S}^+({}^*u_0)$ and if M is the largest positively invariant set contained in*

$$\mathbb{S}^+({}^*u_0) \cap \{f \in {}^*\mathbb{R}^{\Omega_{\mathbb{X}}} : \phi(f(x)) \text{ is constant over } \Omega_{\mathbb{X}}\},$$

then ${}^\lim_{t \rightarrow +\infty} u(t) \in M$. In particular, system 3.6 has at least an asymptotically *stable steady state.*

Proof. This is a consequence of Proposition 2 of [19]. □

We observe that, under the hypothesis that the steady states of system 3.6 are isolated in $\mathbb{S}^+({}^*u_0)$, then **stability* is equivalent to asymptotic **stability*.

Lemma 5.3. *If \tilde{u} is a *stable steady state of system 3.6 and if the steady states are isolated in $\mathbb{S}^+({}^*u_0)$, then \tilde{u} is asymptotically *stable.*

Proof. Suppose that \tilde{u} is **stable*: since the steady states of system 3.6 are isolated, we can find $\rho > 0$ such that if $|\tilde{u} - v| < \rho$ then v is not a steady state of system 3.6. By the **stability* of \tilde{u} , we can find $\delta > 0$ such that if $|\tilde{u} - v| < \delta$ then, denoting by $v(t)$ the solution of system 3.6 with initial data

v , $|\tilde{u} - v(t)| < \rho$ for all $t \in {}^*\mathbb{R}_+$. Moreover, by Proposition 5.2 $v(t)$ converges to a steady state of system 3.6. By our choice of ρ , this steady state must be \tilde{u} , hence \tilde{u} is * attractive. \square

5.2. Steady states of problem 3.6. For a matter of commodity, we will carry out the study of the steady states of problem 3.6 in the case where $k = 1$, and where the spatial domain is $[0, 1]_{\mathbb{X}} = [0, \varepsilon, \dots, N\varepsilon = 1]$, but the analysis can be carried out in higher dimension and with other domains. Moreover, we identify a grid function $u \in {}^*\mathbb{R}^{[0,1]_{\mathbb{X}}}$ with a vector $u \in {}^*\mathbb{R}^{N+1}$, with the convention that u_i , the i -th component of u , satisfies $u_i = u(i\varepsilon)$. If $u : {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}^{[0,1]_{\mathbb{X}}}$, we will identify it with a vector-valued function $u : {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}^{N+1}$, with the convention that $u_i(t) = u(t, i\varepsilon)$.

We begin the study of the * stability of the steady states of system 3.6 by discussing its homogeneous steady state $\tilde{u}_h = (\|{}^*u_0\|_1, \dots, \|{}^*u_0\|_1)$.

Proposition 5.4. *The homogeneous steady state \tilde{u}_h of system 3.6 has the following properties:*

- if $\|{}^*u_0\|_1 < u^-$ or $\|{}^*u_0\|_1 > u^+$, then \tilde{u}_h is * stable;
- if \tilde{u}_h is the only steady state of 3.6, then \tilde{u}_h is globally asymptotically * stable;
- if $u^- < \|{}^*u_0\|_1 < u^+$, then \tilde{u}_h is * unstable. Moreover, if ${}^*u_0 \not\approx \tilde{u}_h$ and if the steady states are isolated in $\mathbb{S}({}^*u_0)$, then u converges to a non-homogeneous steady state.

Proof. It is a consequence of Proposition 3 and of Corollary 4 of [19]. \square

In addition to the homogeneous steady state \tilde{u}_h , system 3.6 may have many non-homogeneous steady states. If we denote by n_i the number of components of \tilde{u} that assume the value ω_i , by Proposition 3.5 we obtain the relations

$$n_3 = N + 1 - (n_1 + n_2), \quad n_1\omega_1 + n_2\omega_2 + (N + 1 - (n_1 + n_2))\omega_3 = (N + 1)\|{}^*u_0\|_1$$

that in the case where $u^+ = +\infty$ become

$$(5.1) \quad n_2 = N + 1 - n_1, \quad n_1\omega_1 + (N + 1 - n_1)\omega_2 = (N + 1)\|{}^*u_0\|_1.$$

In the first step of the study of the * stability of the non-homogeneous steady states of system 3.6, we will prove that all the steady states with $n_2 > 1$ are * unstable.

Proposition 5.5. *If $\tilde{u} \in {}^*\mathbb{R}^{N+1}$ is a steady state of 3.6 with $n_2 > 1$, then it is * unstable.*

Proof. As in the proof of Proposition 4 of Witelski, Schaeffer and Shearer [34], we will show that \tilde{u} is not a stable steady state of 3.6 by showing that it is not a local minimum of a suitable Lyapunov function: the thesis follows from this result. In order to simplify the notation, suppose that $n_3 = 0$, as the proof for the general case can be deduced by the argument below.

Consider the perturbed steady state given by

$$\begin{cases} u_{i_1}(t) &= \omega_2 + q \\ u_{i_2}(t) &= \omega_2 - q \\ u_{i_k}(t) &= \omega_2 & \text{for } k = 3, 4, \dots, n_2 \\ u_i(t) &= \omega_1 & \text{otherwise} \end{cases}$$

Let now $V(u_i) = \int_0^{u_i} \phi(s)ds$, and $L(u) = \sum_{i=0}^{N+1} V(u_i)$. From Proposition 4 of [34], it can be deduced that L is a Lyapunov function for system 3.6. By evaluating L as a function of q , we get

$$L(q) = \frac{V(\omega_2 + q) + V(\omega_2 - q) + (n_2 - 2)V(\omega_2) + (N + 1 - n_2)V(\omega_1)}{N}$$

so we deduce

$$\left. \frac{dL}{dq} \right|_0 = 0 \quad \text{and} \quad \left. \frac{d^2L}{dq^2} \right|_0 = \frac{2}{N} \phi'(q_2) < 0,$$

where the last inequality follows from the hypothesis that $\omega_2 \in (u^-, u^+)$. We conclude that \tilde{u} is not a local minimum of L and, as a consequence, that \tilde{u} is *unstable. \square

The characterization of the asymptotically *stable non-homogeneous steady states of system 3.6 is based on the following bound on $\phi'(q_2)$.

Lemma 5.6. *If \tilde{u} is an asymptotically *stable non-homogeneous steady state of 3.6, then it holds the inequality*

$$(5.2) \quad |\phi'(\omega_2)| < \frac{\max\{\phi'(\omega_1), \phi'(\omega_3)\}^2}{N \min\{\phi'(\omega_1), \phi'(\omega_3)\}}.$$

Proof. For a matter of commodity, suppose that

$$\begin{cases} \tilde{u}(0) &= \omega_2 \\ \tilde{u}(i) &= \omega_1 & \text{for } i = 1, 2, \dots, n_1 \\ \tilde{u}(i) &= \omega_3 & \text{otherwise.} \end{cases}$$

Let $X_1(\tilde{u}) = -(\phi'(\tilde{u}(0)) + \phi'(\tilde{u}(1)))$ and define by recursion

$$X_{i+1}(\tilde{u}) = -\phi'(\tilde{u}(i+1))X_i(\tilde{u}) + (-1)^{i+1} \prod_{j=0}^i \phi'(\tilde{u}(j))$$

It is a consequence of Proposition 8 of [19] that asymptotic *stability of \tilde{u} is equivalent to $(-1)^i X_i(\tilde{u}) > 0$ for $i = 1, \dots, N$. Notice that, as long as $i \leq n_1$,

$$X_i(\tilde{u}) = (-1)^i \phi'(\omega_1)^{i-1} (\phi'(\omega_1) + i\phi'(\omega_2)),$$

so that $(-1)^i X_i(\tilde{u}) > 0$ is equivalent to

$$|\phi'(\omega_2)| < \frac{\phi'(\omega_1)}{i} \leq \frac{\phi'(\omega_1)}{n_1}.$$

For $i = n_1 + 1, \dots, N$, a similar computation shows that $(-1)^i X_i(\tilde{u}) > 0$ implies

$$\begin{aligned} |\phi'(\omega_2)| &< \frac{\phi'(\omega_1)\phi'(\omega_3)}{n_1\phi'(\omega_3) + (i - n_1)\phi'(\omega_1)} \\ &\leq \frac{\phi'(\omega_1)\phi'(\omega_3)}{n_1\phi'(\omega_3) + (N - n_1)\phi'(\omega_1)}. \end{aligned}$$

From the inequality

$$\frac{\phi'(\omega_1)\phi'(\omega_3)}{n_1\phi'(\omega_3) + (N - n_1)\phi'(\omega_1)} \leq \frac{\max\{\phi'(\omega_1), \phi'(\omega_3)\}^2}{N \min\{\phi'(\omega_1), \phi'(\omega_3)\}}$$

we deduce that the desired result holds. \square

5.3. Asymptotic behaviour of the grid solutions under the hypothesis $u^+ < +\infty$. We will now discuss the asymptotic behaviour of the grid solutions to problem 1.1 under the hypothesis that $u^+ < +\infty$. Under this hypothesis, the steady states of the grid function formulation with $n_2 = 0$ are all asymptotically $*$ stable.

Proposition 5.7. *Let \tilde{u} be a steady state of system 3.6 with $n_2 = 0$. Then \tilde{u} is asymptotically $*$ stable.*

Proof. It is a consequence of Proposition 8 of [19]. \square

It turns out that, thanks to the hypotheses over ϕ , all $*$ stable non-homogeneous steady states of system 3.6 for which $\omega_1 \not\approx u^-$ and $\omega_3 \not\approx u^+$ must have $n_2 = 0$, giving a partial converse to Proposition 5.7.

Proposition 5.8. *If \tilde{u} is an asymptotically $*$ stable non-homogeneous steady state of 3.6 with $\omega_1 \not\approx u^-$ and $\omega_3 \not\approx u^+$, then $n_2 = 0$.*

Proof. Suppose towards a contradiction that $n_2 = 1$. The hypotheses $\omega_1 \not\approx u^-$ and $\omega_3 \not\approx u^+$ imply $\min\{*\phi'(\omega_1), *\phi'(\omega_3)\} \not\approx 0$, otherwise either $\phi'(\omega_1) = 0$ or $\phi'(\omega_3) = 0$, against the hypotheses 1.1. As a consequence, $N \min\{*\phi'(\omega_1), *\phi'(\omega_3)\}$ is infinite. Thanks to inequality 5.2, we deduce that $|*\phi'(\omega_2)| \approx 0$. By the hypotheses over ϕ , there exists $\omega_2 \in *(u^-, u^+)$ with $|*\phi'(\omega_2)| \approx 0$ if and only if $\omega_2 \approx u^-$ or $\omega_2 \approx u^+$. However, $\omega_2 \approx u^-$ implies $\omega_1 \approx u^-$ and $\omega_2 \approx u^+$ implies $\omega_3 \approx u^+$, in contradiction with the hypotheses $\omega_1 \not\approx u^-$ and $\omega_3 \not\approx u^+$. \square

Putting together the results of this section, we can characterize the asymptotic behaviour of a grid solution of problem 1.1. In particular, for a.e. initial data, the grid solution converges to a steady state that is a superposition of at most two Dirac measures centred at the stable branches of ϕ .

Proposition 5.9. *Let $[u], [* \phi(u)]$ be the grid solution of problem 1.1 with initial data $*u_0$. For almost every $*u_0 \in L^\infty(\Omega)$, $[u]$ converges to a steady state ν satisfying:*

- (1) *there exists $c \in \mathbb{R}$ such that $\int_{\mathbb{R}} \phi(\tau) d\nu(x) = c$ for all $x \in \Omega_{\mathbb{X}}$;*
- (2) *there exist $\omega_1 \in [0, u^-]$, $\omega_3 \in [u^+, +\infty)$, and $\lambda_1, \lambda_2 : \Omega \rightarrow [0, 1]$, such that*
 - (a) *$\nu(x) = \lambda_1(x)\delta_{\omega_1} + \lambda_2(x)\delta_{\omega_2}$ for a.e. $x \in \Omega$;*
 - (b) *$\phi(\omega_1) = \phi(\omega_2) = c$;*
 - (c) *$\lambda_1(x) + \lambda_2(x) = 1$ for a.e. $x \in \Omega$.*

5.4. Asymptotic behaviour of the grid solutions under the hypothesis $u^+ = +\infty$. If $u^+ = +\infty$, the bound of Lemma 5.6 becomes

$$(5.3) \quad |\phi'(\omega_2)| < \frac{\phi'(\omega_1)}{N}.$$

From this inequality we will deduce that a necessary condition for the asymptotic $*$ stability of a non-homogeneous steady state p is that $\phi'(\omega_2) \approx 0$, and this is possible only when ω_2 is infinite.

Proposition 5.10. *Suppose that $u^+ = +\infty$ and that \tilde{u} is an asymptotically $*$ stable non-homogeneous steady state of 3.6. Then ω_2 is infinite.*

Proof. Since the steady state is non-homogeneous, $\omega_2 > u^-$. By inequality 5.3, and since $\phi \in C^1(\mathbb{R})$, it must hold $\phi'(\omega_2) \approx 0$. Since hypotheses 1.1 entails the inequality $\phi'(x) < 0$ for all $x > u^-$, and this can only happen if ω_2 is infinite, as desired. \square

This result together with Proposition 5.5 implies that any non-homogeneous asymptotically $*$ stable steady states of system 3.6 in the case where $u^+ = +\infty$ are piecewise constant with a single spike. Proposition 5.10 implies that an infinite amount of the mass is concentrated in the spike. This result is in accord with both the theoretical results and the numerical experiments of [14, 15, 19, 27, 34]. However, as we observed previously, we do not expect that $[u]$ converges to a steady state which satisfies Proposition 5.10. Proposition 5.10 should be interpreted as a confirmation of a conjecture by Smarazzo that, for a grid solution $[u]$ of problem 1.1, the regular part of the solution eventually vanishes, and the singular part of the solution prevails. More precisely, we obtain the following result.

Proposition 5.11. *Let $[u], [* \phi(u)]$ be the grid solution of problem 1.1 with initial data $*u_0$. For almost every $*u_0 \in L^\infty(\Omega)$, $[u]$ converges to a steady state $\nu + \mu$ satisfying:*

- (1) *ν is a homogeneous Dirac Young measure centred at 0, i.e. $\nu \in L^\infty(\Omega)$ and $\nu(x) = 0$ a.e.;*
- (2) *$\mu = \|*u_0\|_1 \tilde{\mu}$, and $\tilde{\mu}$ is a probability measure over Ω .*

*In particular, for almost every initial data $*u_0$, $[u]$ converges to a steady state with null regular part.*

6. THE RIEMANN PROBLEM

In the study of problems 1.1 and 1.2 in the case when $u^+ < +\infty$, the dynamics of solutions with Riemann initial data are of particular interest

both in the theoretical and in the numerical setting (see for instance [9, 23]). We will discuss the Riemann problem where the initial data *u_0 satisfies

$$(6.1) \quad {}^*u_0(i\varepsilon) = \begin{cases} \omega_1 \in [0, u^-] & \text{for } 0 \leq i \leq n \\ \omega_3 \in [u^+, +\infty) & \text{for } n+1 \leq i \leq N \end{cases}$$

for some $n \leq N$. In order to understand the evolution of system 3.6 with initial data 6.1, we need to focus on the behaviour of the solution near the discontinuity in the data. In particular, we will discuss the conditions at which $u_i(t) \in (0, u^-)$, $u_{i+1}(t) \in (u^+, +\infty)$ and either $u_i(t + \tau) \in (u^-, u^+)$ or $u_{i+1}(t + \tau) \in (u^-, u^+)$ for some small $\tau > 0$. If $u_i(t) \in (0, u^-)$ and $u_i(t + \tau) \in (u^-, u^+)$ we will say that there is an upward phase transition at $u_i(t)$; if $u_{i+1}(t) \in (u^+, +\infty)$ and $u_{i+1}(t + \tau) \in (u^-, u^+)$ we will say that there is a downward phase transition at $u_{i+1}(t)$.

Proposition 6.1. *Let u be a solution of system 3.6 with initial data 6.1. Then an upward phase transition occurs at $u_i(t)$ for some $t > 0$ and for some $0 \leq i \leq N$ iff $u_i(t) = u^-$,*

$$(6.2) \quad {}^*\phi(u_{i-1}(t)) + {}^*\phi(u_{i+1}(t)) > 2{}^*\phi(u^-)$$

and

$$(6.3) \quad i = \max_{j \in [0, 1]_{\mathbb{X}}} \{j : u_m(t) \leq u^- \text{ for all } m \leq j\}.$$

A downward phase transition occurs at time t at some $0 \leq i \leq N$ iff $u_i(t) = u^+$,

$$(6.4) \quad {}^*\phi(u_{i-1}(t)) + {}^*\phi(u_{i+1}(t)) < 2{}^*\phi(u^+)$$

and

$$(6.5) \quad i = \max_{j \in [0, 1]_{\mathbb{X}}} \{j : u_m(t) \geq u^+ \text{ for all } m \leq j\}.$$

Proof. Suppose that $u_i(t) = u^-$ for some $t \in {}^*\mathbb{R}_+$ and for some $i \leq N$. Then there is a phase transition iff

$$\varepsilon^2 u'_i(t) = {}^*\phi(u_{i-1}(t)) - 2{}^*\phi(u_i(t)) + {}^*\phi(u_{i+1}(t)) > 0$$

from which 6.2 follows. Inequality 6.4 can be proved in a similar way. Notice that the two inequalities imply that if at time t $u_i(t)$, $u_{i+1}(t)$ and $u_{i-1}(t)$ are in the same stable phase, then u_i cannot have a transition at time t . This is sufficient to entail 6.3 and 6.5 for Riemann initial data. \square

Proposition 6.2. *Let u be a solution of system 3.6 with initial data 6.1. For every $t \in {}^*\mathbb{R}_+$, there exists at most one $i \leq N$ such that $u_i(t) \in (u^-, u^+)$.*

Proof. Conditions 6.3 and 6.5 imply that if $u_i(t)$ and $u_{i+1}(t) \in (0, u^-)$ or if $u_i(t)$ and $u_{i+1}(t) \in (u^+, +\infty)$, then they cannot have a simultaneous phase transition. If both $u_i(t)$ and $u_{i+1}(t) \notin (u^-, u^+)$, there cannot be an upwards phase transition at point $u_i(t)$ and a downward phase transition at point $u_{i+1}(t)$: otherwise, from 6.2 and 6.4 we would have ${}^*\phi(u_{i+1}(t)) > {}^*\phi(u^-)$ or ${}^*\phi(u_i(t)) < {}^*\phi(u^+)$, against the necessity that $u_i(t) = u^-$ and $u_{i+1}(t) = u^+$.

If $u_i(t) \in (u^-, u^+)$ and if $u_{i-1}(t)$ had an upwards phase transition, from 6.2 we would have ${}^*\phi(u_{i-2}(t)) > {}^*\phi(u^-)$, contradicting 6.3. If $u_i(t) \in (u^-, u^+)$ and if $u_{i+1}(t)$ had a downward phase transition, from 6.4 we would have ${}^*\phi(u_{i+2}(t)) < {}^*\phi(u^+)$, against 6.5. \square

Notice that Propositions 6.1 and 6.2 can be generalized to any piecewise S-continuous initial data taking values in $(0, u^-) \cup (u^+, +\infty)$: in this case, if the initial data has n discontinuities, then $u_i(t) \in (u^-, u^+)$ for at most n values of $i \leq N$. In particular, if the initial data has finitely many discontinuities, then the dynamics of the system outside of the stable branches of ϕ is negligible. In these cases, it could be argued by the above proposition that the phase transitions of $[u]$ trace a clockwise hysteresis loop, in agreement with the behaviour of two-phase solutions to 1.1 studied in [7, 8, 24].

We conclude our discussion of the Riemann problem with initial data 6.1 with a characterization of the asymptotic behaviour of the solution.

Corollary 6.3. *Let u be the solution of system 3.6 with initial data 6.1. If ${}^*\phi(\omega_1) > {}^*\phi(\omega_2)$ then no phase transitions occur.*

Proof. It is a consequence of 6.2 and 6.4 of Proposition 6.1 and of the fact that ${}^*\phi(\omega_1) > {}^*\phi(\omega_2)$ implies $u'_n(0) < 0$ and $u'_{n+1}(0) > 0$. \square

Corollary 6.4. *Let $[u]$ be the grid solution of problem 1.1 with initial data 6.1. Then $[u]$ converges to an asymptotically stable state that is either constant or Riemann-shaped.*

Proof. If no phase transitions occur, then the thesis is a consequence of Proposition 5.2. If phase transitions occur, this is a consequence of Proposition 5.2 and of Proposition 5.8. \square

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